

ON THE CHARACTERISTIC POLYNOMIAL OF CARTAN MATRICES AND CHEBYSHEV POLYNOMIALS

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ABSTRACT. We explore some interesting features of the characteristic polynomial of the Cartan matrix of a complex simple Lie algebra. The characteristic polynomial is closely related with the Chebyshev polynomials of first and second kind. In addition, we give explicit formulas for the characteristic polynomial of the Coxeter adjacency matrix, we compute the associated polynomials and use them to derive the Coxeter polynomial of the underlying graph. We determine the expression of the Coxeter and associated polynomials as a product of cyclotomic factors. We use this data to propose an algorithm for factoring Chebyshev polynomials over the integers. Finally, we prove an interesting sine formula which involves the exponents, the Coxeter number and the determinant of the Cartan matrix.

1. INTRODUCTION

The aim of this paper is to explore the intimate connection between Chebyshev polynomials and root systems of complex simple Lie algebras. Chebyshev polynomials are used to generate the characteristic and associated polynomials of Cartan and adjacency matrices and conversely one can use machinery from Lie theory to derive properties of Chebyshev polynomials. Some of the results in this paper are well-known but we re-derive them in the context of Chebyshev polynomials.

Cartan matrices appear in the classification of simple Lie algebras over the complex numbers. A Cartan matrix is associated to each such Lie algebra. It is an $\ell \times \ell$ square matrix where ℓ is the rank of the Lie algebra. The Cartan matrix encodes all the properties of the simple Lie algebra it represents. Let \mathfrak{g} be a simple complex Lie algebra, \mathfrak{h} a Cartan subalgebra and $\Pi = \{\alpha_1, \dots, \alpha_\ell\}$ a basis of simple roots for the root system Δ of \mathfrak{h} in \mathfrak{g} . The elements of the Cartan matrix C are given by

$$c_{ij} := 2 \frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$$

where the inner product is induced by the Killing form. The $\ell \times \ell$ -matrix C is invertible. It is called the *Cartan matrix* of \mathfrak{g} . The Cartan matrix for a complex simple Lie algebra obeys the following properties:

- (1) C is symmetrizable. There exists a diagonal matrix D such that DC is symmetric.
- (2) $c_{ii} = 2$.
- (3) $c_{ij} \in \{0, -1, -2, -3\}$ for $i \neq j$.
- (4) $c_{ij} = 0 \Leftrightarrow c_{ji} = 0$.

The complex simple Lie algebras are classified as:

$$A_l, B_l, C_l, D_l, E_6, E_7, E_8, F_4, G_2.$$

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Traditionally, A_l, B_l, C_l, D_l are called the classical Lie algebras while E_6, E_7, E_8, F_4, G_2 are called the exceptional Lie algebras. Moreover, for any Cartan matrix there exists just one simple complex Lie algebra up to isomorphism giving rise to it. The classification of simple complex Lie algebras is due to Killing and Cartan around 1890. According to A. J. Coleman ([4]) the classification paper of Killing is the greatest mathematical writing of all times (after Euclid's Elements and Newton's Principia). Simple Lie algebras over \mathbf{C} are classified by using the associated Dynkin diagram. It is a graph whose vertices correspond to the elements of Π . Each pair of vertices α_i, α_j are connected by

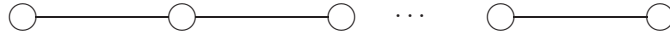
$$m_{ij} = \frac{4(\alpha_i, \alpha_j)^2}{(\alpha_i, \alpha_i)(\alpha_j, \alpha_j)}$$

edges, where

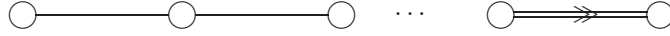
$$m_{ij} \in \{0, 1, 2, 3\} .$$

Dynkin Diagrams for simple Lie algebras

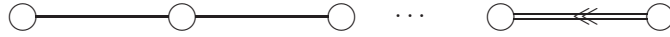
A_n



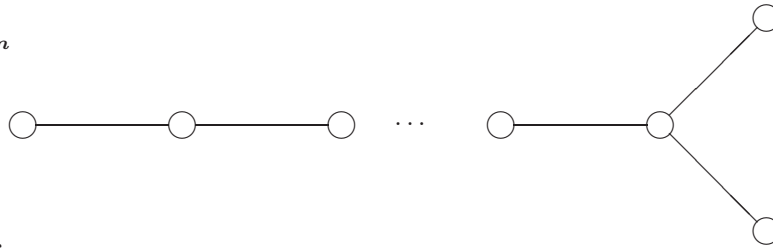
B_n



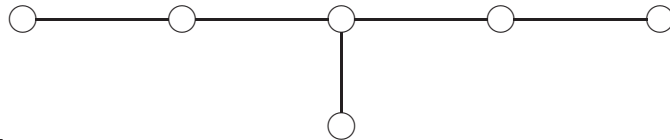
C_n



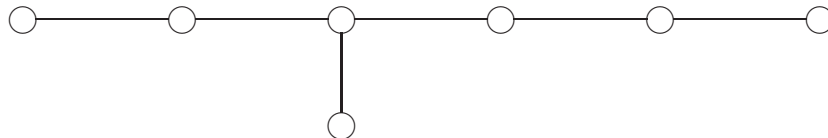
D_n

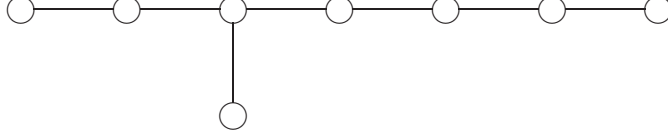
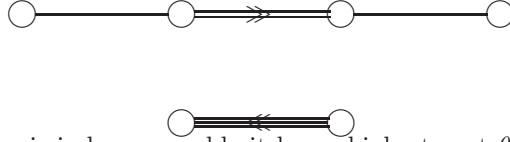


E_6



E_7



E_8  F_4 G_2 

If the root system is indecomposable it has a highest root θ . Let $\alpha_0 = -\theta$ and let $\Pi_0 = \{\alpha_0, \alpha_1, \dots, \alpha_l\}$. As before define

$$\tilde{C} := 2 \frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \quad 0 \leq i, j \leq l .$$

\tilde{C} is called the extended Cartan matrix. It satisfies all the properties of C except that $\det \tilde{C} = 0$. Extended Cartan matrices classify affine Lie algebras. We use the symbols

$$A_l^{(1)}, B_l^{(1)}, C_l^{(1)}, D_l^{(1)}, E_6^{(1)}, E_7^{(1)}, E_8^{(1)}, F_4^{(1)}, G_2^{(1)} ,$$

for the corresponding affine Lie algebras. For more details on these classifications see [13], [14]. The following result due to A. Koranyi [15] is useful in computing the characteristic polynomial of the Cartan matrix.

Theorem 1. *Let C be the $n \times n$ Cartan matrix of a simple Lie algebra over \mathbf{C} . Let $p_n(x)$ be its characteristic polynomial. Then*

$$p_n(x) = q_n \left(\frac{x}{2} - 1 \right)$$

where q_n is a polynomial related to Chebyshev polynomials as follows:

$$\begin{aligned} A_n : \quad & q_n = U_n \\ B_n, C_n : \quad & q_n = 2T_n \\ D_n : \quad & q_n = 4xT_{n-1} \end{aligned}$$

where T_n and U_n are the Chebyshev polynomials of first and second kind respectively.

We have verified similar results for affine Lie algebras:

$$\begin{aligned} A_{n-1}^{(1)} & 2T_n + 2(-1)^{n-1} \\ B_{n-1}^{(1)} & 2(T_n - T_{n-4}) \\ C_{n-1}^{(1)} & 2(T_n - T_{n-2}) \\ D_{n-1}^{(1)} & 8x^2(T_{n-2} - T_{n-4}) . \end{aligned}$$

In this paper we limit our discussion and proofs mainly for the case of simple Lie algebras. The case of affine Lie algebras will be treated in a forthcoming publication.

To a given Dynkin diagram Γ with n nodes we associate the *Coxeter adjacency matrix* which is the $n \times n$ matrix $A = 2I - C$ where C is the Cartan matrix. The

characteristic polynomial of Γ is that of A . Similarly the *norm* of Γ is defined to be the norm of A . One defines the *spectral radius* of Γ to be

$$\rho(\Gamma) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } A\} .$$

If the graph is a tree then the characteristic polynomial p_A of the adjacency matrix is simply related to the characteristic polynomial of the Cartan matrix p_C . In fact,

$$(1) \quad a_n(x) = p_n(x+2) .$$

Using the fact that the spectrum of A is the same as the spectrum of $-A$ it follows easily that if λ is an eigenvalue of the adjacency matrix then $2 + \lambda$ is an eigenvalue of the corresponding Cartan matrix.

In this paper we use the following notation. Note that the subscript n , in all cases, is equal to the degree of the polynomial except in the case of $Q_n(x)$ which is of degree $2n$.

- $p_n(x)$ will denote the characteristic polynomial of the Cartan matrix.
 - $a_n(x)$ will denote the characteristic polynomial of the adjacency matrix.
- Note that

$$a_n(x) = p_n(x+2) = q_n\left(\frac{x}{2}\right) .$$

- Finally we define the associated polynomial

$$Q_n(x) = x^n a_n\left(x + \frac{1}{x}\right) .$$

$Q_n(x)$ turns out to be an even, reciprocal polynomial of the form $Q_n(x) = f_n(x^2)$. The polynomial f_n is the so called Coxeter polynomial of the underlying graph. For the definition and spectral properties of the Coxeter polynomial see, [1], [2], [3], [6], [16], [23], [24]. The roots of Q_n in the cases we consider are in the unit disk and therefore by a theorem of Kronecker, see [8], $Q_n(x)$ is a product of cyclotomic polynomials. We determine the factorization of f_n as a product of cyclotomic polynomials. This factorization in turn determines the factorization of Q_n . The irreducible factors of Q_n are in one-to-one correspondence with the irreducible factors of $a_n(x)$. As a bi-product we obtain the factorization of the Chebyshev polynomials of the first and second kind over the integers. More precisely we prove the following result:

Theorem 2. *Let $\psi_n(x)$ be the minimal polynomial of the algebraic integer $2 \cos \frac{2\pi}{n}$. Then*

$$U_n(x) = \prod_{\substack{j|2n+2 \\ j \neq 1,2}} \psi_j(2x) .$$

Let $n = 2^\alpha N$ where N is odd and let $r = 2^{\alpha+2}$. Then

$$T_n(x) = \frac{1}{2} \prod_{j|N} \psi_{rj}(2x) .$$

The irreducible polynomials ψ_n were introduced by Lehmer in [18]. The factorization is consistent with previous results, e.g. [12], [21].

Using the factorization of the polynomials $a_n(x)$ and $p_n(x)$ we obtain the following interesting sine formula: Let \mathfrak{g} be a complex simple Lie algebra of rank ℓ , h the Coxeter number, m_1, m_2, \dots, m_ℓ the exponents of \mathfrak{g} and C the Cartan matrix. Then

$$2^{2\ell} \prod_i^\ell \sin^2 \frac{m_i \pi}{2h} = \det C .$$

The Mahler measure of a polynomial

$$p(x) = \prod_{k=1}^n (x - \alpha_k)$$

is defined to be

$$M(p) = \prod_{k=1}^n \max \{1, |\alpha_k|\} = \exp \left(\frac{1}{2\pi} \int_0^{2\pi} \ln(|p(e^{i\theta})|) d\theta \right) .$$

According to Kronecker's Theorem $M(p) = 1$ iff $p(x)$ is a product of cyclotomic polynomials and x . D. H. Lehmer posed the following question: Is there a polynomial with Mahler measure between 1 and $1 + \epsilon$ for each $\epsilon > 0$? For more details on Lehmer problem see [11], [19].

A real algebraic integer $\lambda > 1$ is a Salem number if all its conjugate roots have absolute value no greater than 1, and at least one has absolute value 1. It follows that the minimal polynomial of λ is reciprocal. One usually associates to each Salem number a combinatorial object, i.e. a graph. The smallest known Salem number is the largest real root of Lehmer's polynomial

$$(2) \quad l(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$$

which is about 1.1762808. According to Lehmer's Conjecture this is the smallest Salem number. Lehmer's polynomial corresponds to a Dynkin graph of type E_{10} . The associated polynomials corresponding to Dynkin diagrams of simple complex Lie algebras are products of cyclotomic polynomials. In order to produce Salem numbers one has to consider more complicated types of graphs.

2. CHEBYSHEV POLYNOMIALS

To compute explicitly $p_n(x)$ we use the following result due to A. Koranyi [15]:

Proposition 1. *Let C be the $n \times n$ Cartan matrix of a simple Lie algebra over \mathbf{C} . Let $p_n(x)$ be its characteristic polynomial and define $q_n(x) = \det(2xI + A)$. Then*

$$p_n(x) = q_n\left(\frac{x}{2} - 1\right), \quad a_n(x) = q_n\left(\frac{x}{2}\right) .$$

The polynomial q_n is related to Chebyshev polynomials as follows:

$$\begin{aligned} A_n : & \quad q_n = U_n \\ B_n, C_n : & \quad q_n = 2T_n \\ D_n : & \quad q_n = 4xT_{n-1} \end{aligned}$$

where T_n and U_n are the Chebyshev polynomials of first and second kind respectively.

Proof. We give an outline of the proof. Note that

$$\begin{aligned} q_n\left(\frac{x}{2} - 1\right) &= \det \left(2\left(\frac{x-2}{2}\right)I_n + A \right) \\ &= \det (xI_n - 2I_n + A) \\ &= \det (xI_n - 2I_n + 2I_n - C) \\ &= \det (xI_n - C) = p_n(x) . \end{aligned}$$

Furthermore, the matrix A for classical Lie algebras has the form

$$A = \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & & 1 & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots \\ & & & & 1 & 0 & 1 \\ & & & & & 1 & D \end{pmatrix},$$

where D is

$$(0), \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

for the cases A_n, B_n, C_n, D_n respectively. The proof is by induction on n . Suppose the result is proved for n and $n-1$. To get $q_{n+1}(x)$, expand the determinant of $2xI_{n+1} + A$ by the first two rows to obtain

$$q_{n+1}(x) = 2xq_n(x) - q_{n-1}.$$

This is the recurrence relation satisfied by the Chebyshev polynomials T_n and U_n . Sections 3 and 4 cover in detail the cases A_n and B_n respectively. In section 5 we outline the case of D_n .

Finally, note that

$$a_n(x) = p_n(x+2) = q_n\left(\frac{x+2}{2} - 1\right) = q_n\left(\frac{x}{2}\right).$$

□

It is not clear why the Chebyshev polynomials appear. However, it seems that their properties were designed in order to fit nicely the theory of complex simple Lie algebras. In fact, the roots of these polynomials determine the spectrum of the Cartan and adjacency matrices. This information is crucial in the proof of the sine formula. The precise definition and some basic properties of U_n and T_n will be given in sections 3 and 4 respectively.

In the case of exceptional Lie algebras, one can directly compute (preferably using a symbolic manipulation package in the case of E_n) the characteristic polynomials $a_n(x)$ and $p_n(x)$ for each exceptional type. We present the calculations in section 8.

3. CARTAN MATRIX OF TYPE A_n

3.1. Eigenvalues of the A_n Cartan matrix. Toeplitz matrices have constant entries on each diagonal parallel to the main diagonal. Tridiagonal Toeplitz matrices are commonly the result of discretizing differential equations.

The eigenvalues of the Toeplitz matrix

$$(3) \quad \begin{pmatrix} b & a & & & & \\ c & b & & a & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots \\ & & & & c & b & a \\ & & & & & c & b \end{pmatrix}$$

are given by

$$(4) \quad \lambda_j = b + 2a\sqrt{\frac{c}{a}} \cos \frac{j\pi}{n+1} \quad j = 1, 2, \dots, n,$$

TABLE 1. Determinants for Cartan matrices

Lie algebra	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
Det	$n+1$	2	2	4	3	2	1	1	1

see e.g. [22, p. 59].

The Cartan matrix of type A_n is a tri-diagonal matrix of the form.

$$(5) \quad C_{A_n} = \begin{pmatrix} 2 & -1 & & & & & & & \\ -1 & 2 & & & & & & & \\ & & \ddots & \ddots & \ddots & & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & & \ddots & \ddots & \ddots & & \\ & & & & & -1 & 2 & -1 & \\ & & & & & & -1 & 2 & \end{pmatrix}.$$

It appears in the classification theory of simple Lie algebras over \mathbf{C} .

Taking $a = c = -1$, $b = 2$ in (4) we deduce that the eigenvalues of A_n are given by

$$(6) \quad \lambda_j = 2 - 2 \cos \frac{j\pi}{n+1} = 4 \sin^2 \frac{j\pi}{2(n+1)} \quad j = 1, 2, \dots, n.$$

Let d_n be the determinant of C_{A_n} . One can compute it using expansion on the first row and induction. We obtain $d_n = 2d_{n-1} - d_{n-2}$, $d_1 = 2$, $d_2 = 3$. This is a simple linear recurrence with solution $d_n = n+1$.

We conclude that

$$\prod_{j=1}^n 4 \sin^2 \frac{j\pi}{2(n+1)} = n+1.$$

Equivalently

$$(7) \quad 2^{2n} \prod_{j=1}^n \sin^2 \frac{j\pi}{2(n+1)} = n+1. \quad (A_n \text{ sine formula})$$

We refer to this relation as the A_n sine formula.

In table 1 we list the determinants for the Cartan matrices of complex simple Lie algebras.

3.2. The characteristic polynomial. We list the formula for the characteristic polynomial of the matrix A_n for small values of n .

$$\begin{aligned} p_1(x) &= x - 2 \\ p_2(x) &= x^2 - 4x + 3 = (x-1)(x-3) \\ p_3(x) &= x^3 - 6x^2 + 10x - 4 = (x-2)(x^2 - 4x + 2) \\ p_4(x) &= x^4 - 8x^3 + 21x^2 - 20x + 5 = (x^2 - 5x + 5)(x^2 - 3x + 1) \\ p_5(x) &= x^5 - 10x^4 + 36x^3 - 56x^2 + 35x - 6 = (x-1)(x-2)(x-3)(x^2 - 4x + 1) \\ p_6(x) &= x^6 - 12x^5 + 55x^4 - 120x^3 + 126x^2 - 56x + 7 \\ p_7(x) &= x^7 - 14x^6 + 78x^5 - 220x^4 + 330x^3 - 252x^2 + 84x - 8. \end{aligned}$$

Proposition 2. Let $p_n(x)$ be the characteristic polynomial of the Cartan matrix (5). Then

$$p_n(x) = \sum_{j=0}^n (-1)^{n+j} \binom{n+j+1}{2j+1} x^j.$$

We will present the proof of this Proposition in the next subsection 3.3 using properties of Chebyshev polynomials.

3.3. Chebyshev polynomials of the second kind. The Chebyshev polynomials form an infinite sequence of orthogonal polynomials. The Chebyshev polynomial of the second kind of degree n is usually denoted by U_n . We list some properties of Chebyshev polynomials following [20], [25].

A fancy way to define the n th Chebyshev polynomial of the second kind is

$$U_n(x) = \det \begin{pmatrix} 2x & 1 & & & & \\ 1 & 2x & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 2x & 1 \\ & & & & 1 & 2x \end{pmatrix},$$

where n is the size of the matrix. By expanding the determinant with respect to the first row we get the recurrence

$$U_0(x) = 1, \quad U_1(x) = 2x, \quad U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x).$$

It is easy then to compute recursively the first few polynomials:

$$\begin{aligned} U_0(x) &= 1 \\ U_1(x) &= 2x \\ U_2(x) &= 4x^2 - 1 \\ U_3(x) &= 8x^3 - 4x \\ U_4(x) &= 16x^4 - 12x^2 + 1 \\ U_5(x) &= 32x^5 - 32x^3 + 6x \\ U_6(x) &= 64x^6 - 80x^4 + 24x^2 - 1. \end{aligned}$$

Letting $x = \cos \theta$ we obtain

$$U_n(x) = \frac{\sin((n+1)\theta)}{\sin \theta}.$$

$U_n(x)$ is a solution of the differential equation

$$(8) \quad (1 - x^2)y'' - 3xy' + n(n+2)y = 0.$$

There is also an explicit formula which is used in the proof of Proposition 2.

$$(9) \quad U_n(x) = \sum_{j=0}^n (-2)^j \binom{n+j+1}{2j+1} (1-x)^j.$$

Another formula in powers of x is

$$(10) \quad U_n(x) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n-j}{j} (2x)^{n-2j}.$$

The polynomials U_n satisfy the following properties:

$$\begin{aligned} U_n(-x) &= (-1)^n U_n(x) \\ U_n(1) &= n+1 \\ U_{2n}(0) &= (-1)^n \\ U_{2n-1}(0) &= 0. \end{aligned}$$

Knowledge of the roots of U_n implies

$$(11) \quad U_n(x) = 2^n \prod_{j=1}^n \left[x - \cos \left(\frac{j\pi}{n+1} \right) \right] .$$

Setting $x = 1$ in this equation we obtain again the A_n sine formula (7).

Lemma 1.

$$p_n(x) = U_n(x/2 - 1) ,$$

where U_n is the Chebyshev polynomial of the second kind.

Proof. We write the eigenvalue equation in the form $\det(xI_n - C_{A_n}) = 0$, where I_n is the $n \times n$ identity matrix. Explicitly,

$$\det(xI_n - C_{A_n}) = \det \begin{pmatrix} x-2 & 1 & & & & \\ 1 & x-2 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & x-2 & 1 \\ & & & & 1 & x-2 \end{pmatrix} =$$

$$\det \begin{pmatrix} 2\left(\frac{x-2}{2}\right) & 1 & & & & \\ 1 & 2\left(\frac{x-2}{2}\right) & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 2\left(\frac{x-2}{2}\right) & 1 \\ & & & & 1 & 2\left(\frac{x-2}{2}\right) \end{pmatrix} = U_n\left(\frac{x}{2} - 1\right) .$$

□

Remark 1. Note that

$$p_n(0) = U_n(-1) = (-1)^n U_n(1) = (-1)^n (n+1) ,$$

which agrees (up to a sign) with the formula for the determinant of A_n . Also

$$p_n(2) = U_n(0) = 0$$

if n is odd. Therefore for n odd, $p_n(x)$ is divisible by $x - 2$.

We can now prove Proposition 2:

Proof. We use the notation

$$c_{nj} = \binom{n+j+1}{2j+1} .$$

$$\begin{aligned} (-1)^n p_n(x) &= (-1)^n U_n\left(\frac{x}{2} - 1\right) = U_n\left(1 - \frac{x}{2}\right) = \\ &= \sum_{j=0}^n (-2)^j c_{nj} \left(1 - \left(1 - \frac{x}{2}\right)\right)^j \\ &= \sum_{j=0}^n (-2)^j c_{nj} \left(\frac{x}{2}\right)^j = \sum_{j=0}^n (-1)^j c_{nj} x^j . \end{aligned}$$

Therefore

$$p_n(x) = \sum_{j=0}^n (-1)^{n+j} c_{nj} x^j .$$

□

4. CARTAN MATRIX OF TYPE B_n AND C_n

4.1. Chebyshev polynomials of the first kind. The Chebyshev polynomials of the first kind are denoted by $T_n(x)$.

They are defined by the recurrence

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) .$$

Using this recursion one may compute the first few polynomials:

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_2(x) &= 2x^2 - 1 \\ T_3(x) &= 4x^3 - 3x \\ T_4(x) &= 8x^4 - 8x^2 + 1 \\ T_5(x) &= 16x^5 - 20x^3 + 5x \\ T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1 . \end{aligned}$$

It is well-known that $\cos(n\theta)$ can be expressed as a polynomial in $\cos(\theta)$. For example

$$\begin{aligned} \cos(0\theta) &= 1 \\ \cos(1\theta) &= \cos \theta \\ \cos(2\theta) &= 2(\cos \theta)^2 - 1 \\ \cos(3\theta) &= 4(\cos \theta)^3 - 3(\cos \theta) . \end{aligned}$$

More generally we have:

$$\cos(n\theta) = T_n(\cos \theta) .$$

$T_n(x)$ is a solution of the differential equation

$$(12) \quad (1 - x^2)y'' - xy' + n^2y = 0 .$$

There is also an explicit formula which is used in the proof of Propositions 3, 4.

$$(13) \quad T_n(x) = n \sum_{j=0}^n (-2)^j \frac{(n+j-1)!}{(n-j)!(2j)!} (1-x)^j \quad (n > 0) .$$

Another formula in powers of x is

$$(14) \quad T_n(x) = \frac{n}{2} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \frac{(n-j-1)!}{j!(n-2j)!} (2x)^{n-2j} .$$

The polynomials T_n satisfy the following properties:

$$\begin{aligned} T_n(-x) &= (-1)^n T_n(x) \\ T_n(1) &= 1 \\ T_{2n}(0) &= (-1)^n \\ T_{2n-1}(0) &= 0 . \end{aligned}$$

Also

$$(15) \quad T_n(x) = 2^{n-1} \prod_{j=1}^n \left[x - \cos \left(\frac{(2j-1)\pi}{2n} \right) \right] .$$

Setting $x = 1$ in this equation quickly leads to the formula

$$(16) \quad 2^{2n} \prod_{j=1}^n \sin^2 \frac{(2j-1)\pi}{4n} = 2 . \quad (B_n \text{ sine formula})$$

We refer to this relation as the B_n sine formula.

We refer to this relation as the B_n sine formula.

4.2. The characteristic polynomial. The Cartan matrix of type B_n is a tri-diagonal matrix of the form

$$(17) \quad C_{B_n} = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & 2 & -1 & \\ & & & -1 & 2 & -2 \\ & & & & -1 & 2 \end{pmatrix}.$$

Since the Cartan matrix of type C_n is the transpose of this matrix we consider only the Cartan matrix of type B_n . Using expansion on the first row it easy to prove that $\det(C_{B_n}) = 2$.

We list the formula for the characteristic polynomial of the matrix B_n for small values of n .

$$\begin{aligned} p_2(x) &= x^2 - 4x + 2 \\ p_3(x) &= x^3 - 6x^2 + 9x - 2 = (x-2)(x^2 - 4x + 1) \\ p_4(x) &= x^4 - 8x^3 + 20x^2 - 16x + 2 \\ p_5(x) &= x^5 - 10x^4 + 35x^3 - 50x^2 + 25x - 2 = (x-2)(x^4 - 8x^3 + 19x^2 - 12x + 1) \\ p_6(x) &= (x^2 - 4x + 1)(x^4 - 8x^3 + 20x^2 - 16x + 1) \\ p_7(x) &= x^7 - 14x^6 + 77x^5 - 210x^4 + 294x^3 - 196x^2 + 49x - 2. \end{aligned}$$

As in the A_n case we define a sequence of polynomials in the following way:

$$q_n(x) = \det \begin{pmatrix} 2x & 1 & & & & \\ 1 & 2x & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & 1 & 2x & 1 & \\ & & & 1 & 2x & 2 \\ & & & & 1 & 2x \end{pmatrix}.$$

By expanding the determinant with respect to the first row we get the recurrence

$$q_1(x) = 2x, \quad q_2(x) = 4x^2 - 2, \quad q_{n+1}(x) = 2xq_n(x) - q_{n-1}(x).$$

We may define $q_0(x) = 2$.

It is easy then to compute recursively the first few polynomials:

$$\begin{aligned} q_0(x) &= 2 \\ q_1(x) &= 2x \\ q_2(x) &= 4x^2 - 2 = 2(2x^2 - 1) \\ q_3(x) &= 8x^3 - 6x = 2(4x^3 - 3) \\ q_4(x) &= 16x^4 - 16x^2 + 2 = 2(8x^4 - 8x^2 + 1) \\ q_5(x) &= 32x^5 - 40x^3 + 10x = 2(16x^5 - 20x^3 + 5x) \\ q_6(x) &= 64x^6 - 96x^4 + 36x^2 - 2 = 2(32x^6 - 48x^4 + 18x^2 - 1) \\ q_7(x) &= 2(64x^7 - 112x^5 + 56x^3 - 7x). \end{aligned}$$

It is clear that $q_n(x) = 2T_n(x)$ where T_n is the n th Chebyshev polynomial of the first kind. Therefore

$$p_n(x) = q_n(x/2 - 1),$$

where $q_n = 2T_n(x)$.

Remark 2. Note that

$$p(0) = q_n(-1) = 2T_n(-1) = (-1)^n 2T_n(1) = 2(-1)^n ,$$

which agrees (up to a sign) with the formula for the determinant of C_{B_n} .

Proposition 3. Let $p_n(x)$ be the characteristic polynomial of the Cartan matrix (17). Then

$$p_n(x) = \sum_{j=0}^{n-1} (-1)^{n+j} \frac{2n(n+j-1)!}{(n-j)!(2j)!} x^j .$$

The proof of this Proposition is along the same lines as the proof of Propositions 2 and 4 and therefore we omit it.

5. CARTAN MATRIX OF TYPE D_n

5.1. The characteristic polynomial. The Cartan matrix of type D_n is a matrix of the form

$$(18) \quad C_{D_n} = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & 2 & -1 & -1 \\ & & & -1 & 2 & 0 \\ & & & & -1 & 0 & 2 \end{pmatrix} .$$

Note that the matrix is no longer tri-diagonal. Using expansion on the first row and induction it easy to prove that $\det(C_{D_n}) = 4$.

We list some formulas for the characteristic polynomial of the matrix C_{D_n} for small values of n .

$$\begin{aligned} p_2(x) &= x^2 - 4x + 2 = (x-2)^2 \\ p_3(x) &= x^3 - 6x^2 + 10x - 4 = (x-2)(x^2 - 4x + 2) \\ p_4(x) &= x^4 - 8x^3 + 21x^2 - 20x + 4 = (x-2)^2(x^2 - 4x + 1) \\ p_5(x) &= x^5 - 10x^4 + 36x^3 - 56x^2 + 34x - 4 = (x-2)(x^4 - 8x^3 + 20x^2 - 16x + 2) \\ p_6(x) &= x^6 - 12x^5 + 55x^4 - 120x^3 + 125x^2 - 52x + 4 = (x-2)^2(x^4 - 8x^3 + 19x^2 - 12x + 1) \end{aligned}$$

Proposition 4. Let $p_n(x)$ be the characteristic polynomial of the Cartan matrix (18). Then

$$p_n(x) = (x-2) \sum_{j=0}^{n-1} (-1)^{n+j} \frac{(2n-2)(n+j-2)!}{(n-j-1)!(2j)!} x^j .$$

Define a sequence of polynomials in the following way:

$$(19) \quad q_n(x) = \det \begin{pmatrix} 2x & 1 & & & & \\ 1 & 2x & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & 1 & 2x & 1 & 1 \\ & & & 1 & 2x & 0 \\ & & & & 1 & 0 & 2x \end{pmatrix} .$$

By expanding the determinant with respect to the first row we get the recurrence

$$q_2(x) = 4x^2, \quad q_3(x) = 8x^3 - 4x, \quad q_{n+1}(x) = 2xq_n(x) - q_{n-1} .$$

We may define $q_1(x) = 4x$. It produces the following sequence of polynomials:

$$\begin{aligned}
q_1(x) &= 4x \\
q_2(x) &= 4x^2 \\
q_3(x) &= 8x^3 - 4x = 4x(2x^2 - 1) \\
q_4(x) &= 16x^4 - 12x^2 = 4x^2(4x^2 - 3) \\
q_5(x) &= 32x^5 - 32x^3 + 4x = 4x(8x^4 - 8x^2 + 1) \\
q_6(x) &= 64x^6 - 80x^4 + 20x^2 = 4x^2(16x^4 - 20x^2 + 5) \\
q_7(x) &= 128x^7 - 192x^5 + 72x^3 - 4x = 4x(2x^2 - 1)(16x^4 - 16x^2 + 1) .
\end{aligned}$$

It is clear that $q_n(x) = 4xT_{n-1}(x)$ where T_n is the n th Chebyshev polynomial of the first kind. Equivalently, $q_n = 2(T_n + T_{n-2})$. As in the previous cases, we conclude that

$$p_n(x) = q_n(x/2 - 1) .$$

where $q_n(x) = 4xT_{n-1}(x)$.

Note that

$$p(0) = q_n(-1) = (-1)^n q_n(1) = (-1)^n 4T_{n-1}(1) = 4(-1)^n ,$$

which agrees with the formula for the determinant of D_n .

We now give the proof of Proposition 4:

Proof. Let $p_n(x)$ be the characteristic polynomial of the Cartan matrix of type D_n . We use the notation

$$c_{nj} = \frac{(n+j-2)!}{(n-j-1)!(2j)!} ,$$

and

$$d_{nj} = \frac{(2n-2)(n+j-2)!}{(n-j-1)!(2j)!}$$

$$(-1)^n p_n(x) = (-1)^n q_n\left(\frac{x}{2} - 1\right) = q_n\left(1 - \frac{x}{2}\right) =$$

$$= 2(x-2)T_{n-1}\left(1 - \frac{x}{2}\right) =$$

$$2(x-2)(n-1) \sum_{j=0}^{n-1} (-2)^j c_{nj} \left(1 - \left(1 - \frac{x}{2}\right)\right)^j$$

$$= 2(x-2)(n-1) \sum_{j=0}^{n-1} (-2)^j c_{nj} \left(\frac{x}{2}\right)^j$$

$$= (x-2) \sum_{j=0}^{n-1} (-1)^j d_{nj} x^j .$$

Therefore

$$p_n(x) = (x-2) \sum_{j=0}^{n-1} (-1)^{n+j} d_{nj} x^j .$$

□

6. THE COXETER POLYNOMIAL

A Coxeter graph is a simple graph Γ with n vertices and edge weights $m_{ij} \in \{3, 4, \dots, \infty\}$. We define $m_{ii} = 1$ and $m_{ij} = 2$ if node i is not connected with node j . By convention if $m_{ij} = 3$ then the edge is often not labeled. If Γ is a Coxeter graph with n vertices we define a bilinear form B on \mathbf{R}^n by choosing a basis e_1, e_2, \dots, e_n and setting

$$B(e_i, e_j) = -2 \cos \frac{\pi}{m_{ij}} .$$

If $m_{ij} = \infty$ we define $B(e_i, e_j) = -2$. We also define for $i = 1, 2, \dots, n$ the reflection

$$\sigma_i(e_j) = e_j - B(e_i, e_j)e_i .$$

Let $S = \{\sigma_i \mid i = 1, \dots, n\}$. The Coxeter group $W(\Gamma)$ is the group generated by the reflections in S . W has the presentation

$$W = \langle \sigma_1, \sigma_2, \dots, \sigma_n \mid \sigma_i^2 = 1, (\sigma_i \sigma_j)^{m_{ij}} = 1 \rangle .$$

It is well-known that $W(\Gamma)$ is finite if and only if B is positive definite. A *Coxeter element* (or transformation) is a product of the form

$$\sigma_{\alpha(1)} \sigma_{\alpha(2)} \dots \sigma_{\alpha(n)} \quad \alpha \in S_n .$$

If the Coxeter graph is a tree then the Coxeter elements are in a single conjugacy class in W . A *Coxeter polynomial* for the Coxeter system (W, S) is the characteristic polynomial of the matrix representation of a Coxeter element. For Coxeter systems whose graphs are trees the Coxeter polynomial is uniquely determined. This covers all the cases we investigate. We define $\rho(W)$ to be the spectral radius of the associated Coxeter adjacency matrix. A Coxeter system is called

- (1) Spherical if $\rho(W) < 2$
- (2) Affine if $\rho(W) = 2$
- (3) Hyperbolic or higher-rank if $\rho(W) > 2$.

In this paper we consider only spherical Coxeter systems. In this case B is positive definite and the Coxeter element has finite order.

Example 1. Consider a Coxeter system with graph A_3 .

The bilinear form is defined by the Cartan matrix

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} .$$

The reflection σ_1 is determined by the action

$$\sigma_1(e_1) = -e_1, \quad \sigma_1(e_2) = e_1 + e_2, \quad \sigma_1(e_3) = e_3 .$$

It has the matrix representation

$$\sigma_1 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

Similarly

$$\sigma_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} .$$

TABLE 2. Exponents and Coxeter number for root systems.

Root system	Exponents	Coxeter number
\mathcal{A}_n	$1, 2, 3, \dots, n$	$n + 1$
\mathcal{B}_n	$1, 3, 5, \dots, 2n - 1$	$2n$
\mathcal{C}_n	$1, 3, 5, \dots, 2n - 1$	$2n$
\mathcal{D}_n	$1, 3, 5, \dots, 2n - 3, n - 1$	$2n - 2$
\mathcal{E}_6	$1, 4, 5, 7, 8, 11$	12
\mathcal{E}_7	$1, 5, 7, 9, 11, 13, 17$	18
\mathcal{E}_8	$1, 7, 11, 13, 17, 19, 23, 29$	30
\mathcal{F}_4	$1, 5, 7, 11$	12
\mathcal{G}_2	$1, 5$	6

A Coxeter element is defined by

$$R = \sigma_1 \sigma_2 \sigma_3 = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix} .$$

The Coxeter polynomial is the characteristic polynomial of R which is

$$x^3 + x^2 + x + 1 .$$

Note that R has order 4, $R^4 = I$. The order of the Coxeter element is called the Coxeter number. Note that the roots of the Coxeter polynomial are $-1, -i, i$. We can write $i = i^1, -1 = i^2, -i = i^3$. The integers 2, 3, 4 are the degrees of the Chevalley invariants and their product, 24, is the order of the Coxeter group.

In general the order of the Coxeter group is

$$(m_1 + 1)(m_2 + 1) \dots (m_l + 1)$$

where m_i are the exponents of the eigenvalues of the Coxeter polynomial. The factors $m_i + 1$ are the degrees of the Chevalley invariants. If ζ is a primitive h root of unity (where h is the Coxeter number) then the roots of the Coxeter polynomial are ζ^m where m runs over the exponents of the corresponding root system [4], [7]. This observation allows the calculation of the Coxeter polynomial for each root system.

In table 2 we list the Coxeter number and the exponents for each root system.

6.1. Exponents. Let us recall the definition of exponents for a simple complex Lie group G , see [3], [5], [17]. Suppose G is a connected, complex, simple Lie Group G . We form the de Rham cohomology groups $H^i(G, \mathbf{C})$ and the corresponding Poincaré polynomial of G :

$$p_G(t) = \sum_{i=1}^{\ell} b_i t^i ,$$

where $b_i = \dim H^i(G, \mathbf{C})$ are the Betti numbers of G . The De Rham groups encode topological information about G . Following work of Cartan, Ponrjagin and Brauer, Hopf proved that the cohomology algebra is isomorphic to that of a finite product of ℓ spheres of odd dimension where ℓ is the rank of G . This result implies that

$$p_G(t) = \prod_{i=1}^{\ell} (1 + t^{2m_i+1}) .$$

The positive integers $\{m_1, m_2, \dots, m_{\ell}\}$ are called the *exponents* of G . They are also the exponents of the Lie algebra \mathfrak{g} of G . One can also extract the exponents from the root space decomposition of \mathfrak{g} following methods which were developed

by R. Bott, A. Shapiro, A.J. Coleman and B. Kostant. The exponents of a simple complex Lie algebra are given in table 2.

Note the duality in the set of exponents:

$$(20) \quad m_i + m_{\ell+1-i} = h$$

where h is the Coxeter number.

6.2. Cyclotomic polynomials. A complex number ω is called a *primitive n th root of unity* provided ω is an n th root of unity and has order n . Such an ω generates the group of units, i.e., $\omega, \omega^2, \dots, \omega^n = 1$ coincides with the set of all roots of unity. An n th root of 1 of the form ω^k is a primitive root of unity iff k is relatively prime to n . Therefore the number of primitive roots of unity is equal to $\phi(n)$ where ϕ is Euler's totient function.

We define the n th cyclotomic polynomial by

$$\Phi_n(x) = (x - \omega_1)(x - \omega_2) \cdots (x - \omega_s) ,$$

where $\omega_1, \omega_2, \dots, \omega_s$ are all the distinct primitive n th roots of unity. The degree of Φ_n is of course equal to $s = \phi(n)$. $\Phi_n(x)$ is a monic, irreducible polynomial with integer coefficients. The first twenty cyclotomic polynomials are given below:

$$\begin{aligned} \Phi_1(x) &= x - 1 \\ \Phi_2(x) &= x + 1 \\ \Phi_3(x) &= x^2 + x + 1 \\ \Phi_4(x) &= x^2 + 1 = \Phi_2(x^2) \\ \Phi_5(x) &= x^4 + x^3 + x^2 + x + 1 \\ \Phi_6(x) &= x^2 - x + 1 = \Phi_3(-x) \\ \Phi_7(x) &= x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \\ \Phi_8(x) &= x^4 + 1 = \Phi_2(x^4) = \Phi_4(x^2) \\ \Phi_9(x) &= x^6 + x^3 + 1 = \Phi_3(x^3) \\ \Phi_{10}(x) &= x^4 - x^3 + x^2 - x + 1 = \Phi_5(-x) \\ \Phi_{11}(x) &= x^{10} + x^9 + \cdots + x + 1 \\ \Phi_{12}(x) &= x^4 - x^2 + 1 = \Phi_6(x^2) \\ \Phi_{13}(x) &= x^{12} + x^{11} + \cdots + x + 1 \\ \Phi_{14}(x) &= x^6 - x^5 + x^4 - x^3 + x^2 - x + 1 = \Phi_7(-x) \\ \Phi_{15}(x) &= x^8 - x^7 + x^5 - x^4 + x^3 - x + 1 \\ \Phi_{16}(x) &= x^8 + 1 = \Phi_2(x^8) = \Phi_8(x^2) \\ \Phi_{17}(x) &= x^{16} + x^{15} + \cdots + x + 1 \\ \Phi_{18}(x) &= x^6 - x^3 + 1 = \Phi_9(-x) \\ \Phi_{19}(x) &= x^{18} + x^{17} + \cdots + x + 1 \\ \Phi_{20}(x) &= x^8 - x^6 + x^4 - x^2 + 1 = \Phi_{10}(x^2) . \end{aligned}$$

A basic formula for cyclotomic polynomials is

$$(21) \quad x^n - 1 = \prod_{d|n} \Phi_d(x) ,$$

where d ranges over all positive divisors of n . This gives a recursive method of calculating cyclotomic polynomials. For example, if $n = p$, where p is prime, then $x^p - 1 = \Phi_1(x)\Phi_p(x)$ which implies that

$$\Phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1 .$$

In table 3 we list the Coxeter polynomial and its factorization as a product of cyclotomic polynomials. The cases B_n and D_n are explained in remark 3.

TABLE 3. Coxeter polynomials for Spherical Graphs

Dynkin Diagram	Coxeter polynomial	Cyclotomic Factors
\mathcal{A}_n	$x^n + x^{n-1} + \dots x + 1$	$\prod_{d (n+1), d \neq 1} \Phi_d$
$\mathcal{B}_n, \mathcal{C}_n$	$x^n + 1$	$\prod_{d N} \Phi_{2md}(x)(*)$
\mathcal{D}_n	$x^n + x^{n-1} + x + 1$	$\Phi_2 \prod_{d N} \Phi_{2md}(x)(**)$
\mathcal{E}_6	$x^6 + x^5 - x^3 + x + 1$	$\Phi_3 \Phi_{12}$
\mathcal{E}_7	$x^7 + x^6 - x^4 - x^3 + x + 1$	$\Phi_2 \Phi_{18}$
\mathcal{E}_8	$x^8 + x^7 - x^5 - x^4 - x^3 + x + 1$	Φ_{30}
\mathcal{F}_4	$x^4 - x^2 + 1$	Φ_{12}
\mathcal{G}_2	$x^2 - x + 1$	Φ_6

Remark 3. (*) $x^n + 1$ is irreducible iff n is a power of 2, see [9]. If n is odd it has a factor of $x + 1 = \Phi_2(x)$. To obtain the factorization of $x^n + 1$ we use the following procedure. Let $n = 2^\alpha p_1^{k_1} p_2^{k_2} \dots p_s^{k_s}$ where the p_i are odd primes. Define $m = 2^\alpha$ and $N = p_1^{k_1} \dots p_s^{k_s}$. The factorization of $x^n + 1$ is given by

$$x^n + 1 = \prod_{d|N} \Phi_{2md}(x) .$$

(**) In the case of D_n we use the same procedure with n replaced by $n - 1$.

Remark 4. In table 3 note the following: In the factorization of $f_n(x)$, the factor of highest degree is Φ_h where h is the Coxeter number.

The roots of $U_n(x)$ are given by

$$x_k = \cos \left(\frac{k\pi}{n+1} \right) \quad k = 1, 2, \dots, n .$$

Therefore the roots of

$$a_n(x) = U_n \left(\frac{x}{2} \right)$$

are in the interval $[-2, 2]$. If a monic polynomial with integer coefficients has all of its roots in the interval $[-2, 2]$ then they are of a special form. We follow the proof from [10].

Proposition 5. Let λ be a non-zero real root of a monic polynomial $p(x) \in \mathbf{Z}[x]$. If all the roots of $p(x)$ are real and in the interval $[-2, 2]$ then $\lambda = 2 \cos(2\pi q)$ where q is a rational number.

Proof. Let $n = \deg(p)$ and define the associated polynomial

$$Q(x) = x^n p(x + \frac{1}{x}) .$$

Denote by

$$\lambda = 2 \cos \theta_1, 2 \cos \theta_2, \dots, 2 \cos \theta_n$$

the roots of $p(x)$. Then

$$p(x) = \prod_j (x - 2 \cos \theta_j)$$

$$Q(x) = \prod_j (x^2 - 2x \cos \theta_j + 1) = \prod_j (x - e^{i\theta_j})(x - e^{-i\theta_j})$$

It follows from Kronecker's Theorem, see [8], that $e^{i\theta_1}$ is a root of unity and therefore $\frac{\theta_1}{2\pi}$ is rational. \square

Example 2. Let us consider $U_n(\frac{x}{2})$ for $n = 5$. The polynomial factors as

$$a_5(x) = U_5(\frac{x}{2}) = x^5 - 4x^3 + 3x = x(x-1)(x+1)(x^2-3)$$

and the roots are $-\sqrt{3}, -1, 0, 1, \sqrt{3}$. As in the proof of the Proposition 5 we can write them as

$$2 \cos \frac{\pi}{6} \quad 2 \cos \frac{\pi}{3} \quad 2 \cos \frac{\pi}{2} \quad 2 \cos \frac{2\pi}{3} \quad 2 \cos \frac{5\pi}{6} .$$

The associated polynomial similarly factors as

$$Q_5(x) = x^{10} + x^8 + x^4 + x^2 + 1 = (x^2 + x + 1)(x^2 + 1)(x^2 - x + 1)(x^4 - x^2 + 1) .$$

$$Q_5(x) = \Phi_3 \Phi_4 \Phi_6 \Phi_{12} .$$

The largest eigenvalue is $2 \cos \frac{\pi}{6}$. In general, and for each simple, complex, Lie algebra, the maximum eigenvalue is

$$2 \cos \frac{\pi}{h}$$

where h is the Coxeter Number and the roots of $a_n(x)$ are $2 \cos \frac{m_i \pi}{h}$ where m_i is an exponent of \mathfrak{g} .

Given a graph Γ we define the associated polynomial $Q_n(x)$ by the formula

$$Q_n(x) = x^n a_n(x + \frac{1}{x}) ,$$

where $a_n(x)$ is the characteristic polynomial of the graph. It is clear that $Q_n(x)$ is a reciprocal polynomial. In all the cases we consider $Q_n(x)$ is an even polynomial.

6.3. Associated polynomials for A_n . Using the same procedure as in the example, we present the factorization of $Q_n(x)$ for small values of n as a product of cyclotomic polynomials. Note that $Q_n(x)$ is an even polynomial. The reason: If n is even $U_n(x)$ is an even polynomial and $a_n(x) = U_n(\frac{x}{2})$ is also even. Therefore $Q_n(x)$ is even. If n is odd then $U_n(x)$ and $a_n(x)$ are both odd functions. This implies that $Q_n(x)$ is even.

- A_2 $\Phi_3 \Phi_6$
- A_3 $\Phi_4 \Phi_8$
- A_4 $\Phi_5 \Phi_{10}$
- A_5 $\Phi_3 \Phi_4 \Phi_6 \Phi_{12}$
- A_6 $\Phi_7 \Phi_{14}$
- A_7 $\Phi_4 \Phi_8 \Phi_{16}$
- A_8 $\Phi_3 \Phi_6 \Phi_9 \Phi_{18}$
- A_9 $\Phi_4 \Phi_5 \Phi_{10} \Phi_{20}$
- A_{10} $\Phi_{11} \Phi_{22}$
- A_{11} $\Phi_3 \Phi_4 \Phi_6 \Phi_8 \Phi_{12} \Phi_{24}$

The characteristic polynomial of the Coxeter transformation has roots ζ^k where ζ is a primitive h root of unity and k runs over the exponents of a root system of type A_n . Therefore

$$\begin{aligned} f_n(x) &= (x - \zeta)(x - \zeta^2) \dots (x - \zeta^n) . \\ \Rightarrow (x - 1)f_n(x) &= x^{n+1} - 1 \\ \Rightarrow f_n(x) &= \frac{x^{n+1} - 1}{x - 1} . \end{aligned}$$

Using formula (21) we obtain

$$f_n(x) = \prod_{\substack{d|n+1 \\ d \neq 1}} \Phi_d .$$

It is not difficult to guess the factorization of $Q_n(x)$.

Proposition 6.

$$(22) \quad Q_n(x) = \prod_{\substack{j|2n+2 \\ j \neq 1,2}} \Phi_j(x) .$$

Proof. Since

$$Q_n(x) = f_n(x^2) = \prod_{\substack{j|n+1 \\ j \neq 1}} \Phi_j(x^2)$$

we should know what is $\Phi_j(x^2)$.

It is well-known, see [9], that

$$\Phi_j(x^2) = \begin{cases} \Phi_{2j}(x), & \text{if } j \text{ is even} \\ \Phi_j(x)\Phi_{2j}(x), & \text{if } j \text{ is odd} \end{cases}$$

To complete the proof we must show that each divisor of $2n+2$ bigger than 2 appears in the product (22). Let d be a divisor of $2n+2$ bigger than 2. We consider two cases:

i) If d is odd then since $d|2(n+1)$ we have that $d|n+1$. Since Φ_d is a factor of $f_n(x)$, then $f_d(x^2) = \Phi_d(x)\Phi_{2d}(x)$, and therefore Φ_d appears.

ii) If d is even, then $d = 2s$ for some integer s bigger than 1. Since $2s|2(n+1)$ we have that $s|n+1$. Therefore Φ_s appears in the factorization of $f_n(x)$. If s is odd then $\Phi_s(x^2) = \Phi_s(x)\Phi_{2s}(x)$ and if s is even $\Phi_s(x^2) = \Phi_{2s}(x)$. In either case $\Phi_{2s} = \Phi_d$ appears. □

An alternative way to derive the formula for f_{A_n} is the following: In the case of A_n we have

$$a_n(x) = U_n\left(\frac{x}{2}\right) .$$

Therefore,

$$Q_n(x) = x^n U_n\left(\frac{1}{2}\left(x + \frac{1}{x}\right)\right) .$$

Set $x = e^{i\theta}$ to obtain

$$\begin{aligned} Q_n(x) &= e^{in\theta} U_n\left(\frac{1}{2}(e^{i\theta} + e^{-i\theta})\right) \\ &= e^{in\theta} U_n(\cos \theta) \\ &= e^{in\theta} \frac{\sin(n+1)\theta}{\sin \theta} \\ &= e^{in\theta} \frac{(e^{i(n+1)\theta} + e^{-i(n+1)\theta})}{e^{i\theta} - e^{-i\theta}} \\ &= \frac{x^{2(n+1)} - 1}{x^2 - 1} . \end{aligned}$$

Setting $u = x^2$ we obtain

$$\frac{u^{n+1} - 1}{u - 1} = u^n + u^{n-1} + \cdots + u + 1 .$$

This is of course the Coxeter polynomial of the A_n graph. Therefore $Q_n(x) = x^{2n} + x^{2(n-1)} + \cdots + x^2 + 1$ for all $x \in \mathbf{C}$.

We present the characteristic polynomial of the adjacency matrix and the Coxeter polynomial for small values of n .

- A_2 $a_2 := x^2 - 1$ $f_2 = \Phi_3$
- A_3 $a_3 := x^3 - 2x$ $f_3 = \Phi_2\Phi_4$
- A_4 $a_4 := x^4 - 3x^2 + 1$ $f_4 = \Phi_5$
- A_5 $a_5 := x^5 - 4x^3 + 3x$ $f_5 = \Phi_2\Phi_3\Phi_6$
- A_6 $a_6 := x^6 - 5x^4 + 6x^2 - 1$ $f_6 = \Phi_7$
- A_7 $a_7 := x^7 - 6x^5 + 10x^3 - 4x$ $f_7 = \Phi_2\Phi_4\Phi_8$
- A_8 $a_8 := x^8 - 7x^6 + 15x^4 - 10x^2 + 1$ $f_8 = \Phi_3\Phi_9$
- A_9 $a_9 := x^9 - 8x^7 + 21x^5 - 20x^3 + 5x$ $f_9 = \Phi_2\Phi_5\Phi_{10}$
- A_{10} $a_{10} := x^{10} - 9x^8 + 26x^6 - 35x^4 + 15x^2 - 1$ $f_{10} = \Phi_{11}$.

Note that $a_n(x)$ is explicitly given by the formula

$$a_n(x) = U_n\left(\frac{x}{2}\right) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n-j}{j} (x)^{n-2j}$$

due to (10).

6.4. Associated Polynomials for B_n and C_n . In the case of B_n we have

$$a_n(x) = 2T_n\left(\frac{x}{2}\right) .$$

Therefore,

$$Q_n(x) = 2x^n T_n\left(\frac{1}{2}\left(x + \frac{1}{x}\right)\right) .$$

Set $x = e^{i\theta}$ to obtain

$$\begin{aligned} Q_n(x) &= 2e^{in\theta} T_n\left(\frac{1}{2}(e^{i\theta} + e^{-i\theta})\right) \\ &= 2e^{in\theta} T_n(\cos \theta) \\ &= 2e^{in\theta} \cos n\theta \\ &= 2e^{in\theta} \frac{1}{2} (e^{in\theta} + e^{-in\theta}) = e^{2in\theta} + 1 = x^{2n} + 1 . \end{aligned}$$

Therefore $Q_n(x) = x^{2n} + 1$ for all $x \in \mathbf{C}$. As a result the Coxeter polynomial is $f_n(x) = x^n + 1$. We present the factorization of $f_n(x)$ for small values of n .

- B_2 $a_2 = x^2 - 2$ $f_2 = \Phi_4$
- B_3 $a_3 = x^3 - 3x$ $f_3 = \Phi_2\Phi_6$
- B_4 $a_4 = x^4 - 4x^2 + 2$ $f_4 = \Phi_8$
- B_5 $a_5 = x^5 - 5x^3 + 5x$ $f_5 = \Phi_2\Phi_{10}$
- B_6 $a_6 = x^6 - 6x^4 + 9x^2 - 2$ $f_6 = \Phi_4\Phi_{12}$
- B_7 $a_7 = x^7 - 7x^5 + 14x^3 - 7x$ $f_7 = \Phi_2\Phi_{14}$
- B_8 $a_8 = x^8 - 8x^6 + 20x^4 - 16x^2 + 2$ $f_8 = \Phi_{16}$
- B_9 $a_9 = x^9 - 9x^7 + 27x^5 - 30x^3 + 9x$ $f_9 = \Phi_2\Phi_6\Phi_{18}$
- B_{10} $a_{10} = x^{10} - 10x^8 + 35x^6 - 50x^4 + 25x^2 - 2$ $f_{10} = \Phi_4\Phi_{20}$.

Write $n = 2^\alpha N$ where N is odd. As we already mentioned

$$f_n(x) = x^n + 1 = \prod_{d|N} \Phi_{2md}(x) ,$$

where $m = 2^\alpha$. Therefore

$$f_n(x) = x^n + 1 = \prod_{\substack{d|n \\ d \text{ odd}}} \Phi_{2^{\alpha+1}d}(x) = \prod_{d|N} \Phi_{2^{\alpha+1}d}(x) .$$

Proposition 7. *Let $r = 2^{\alpha+2}$. Then*

$$(23) \quad Q_n(x) = \prod_{\substack{d|n \\ d \text{ odd}}} \Phi_{rd}(x) .$$

Proof. It follows from the formula

$$\Phi_k(x^2) = \Phi_{2k}(x)$$

when k is even. □

Example 3. *Let $n = 24$. Then $24 = 2^3 \cdot 3$. Therefore $\alpha = 3$.*

$$Q_{24}(x) = x^{48} + 1 = \prod_{\substack{d|24 \\ d \text{ odd}}} \Phi_{32d} = \Phi_{32} \Phi_{96} = (x^{16} + 1)(x^{32} - x^{16} + 1) .$$

Note that the $a_n(x)$ polynomial is explicitly given in this case by the formula

$$a_n(x) = 2T_n\left(\frac{x}{2}\right) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \frac{n(n-j-1)!}{j!(n-2j)!} (x)^{n-2j}$$

due to (14).

Since $a_n(x) = 2T_n(\frac{x}{2})$ these polynomials satisfy the recursion

$$a_{n+1} = xa_n(x) - a_{n-1}(x)$$

with $a_0(x) = 2$ and $a_1(x) = x$. We would like to mention a useful application of these polynomials. One can use them to express $x^n + x^{-n}$ as a function of $\zeta = x + \frac{1}{x}$. For $x = e^{i\theta}$ it is just the expression of $2 \cos n\theta$ as a polynomial in $2 \cos \theta$. This polynomial is clearly $a_n(x)$, the adjacency polynomial of B_n .

Example 4.

$$\left(x + \frac{1}{x}\right)^2 = x^2 + \frac{1}{x^2} + 2 .$$

Therefore

$$x^2 + \frac{1}{x^2} = \zeta^2 - 2 = a_2(\zeta) .$$

Similarly

$$x^3 + \frac{1}{x^3} = \zeta^3 - 3\zeta = a_3(\zeta) ,$$

$$x^4 + \frac{1}{x^4} = \zeta^4 - 4\zeta^2 + 2 = a_4(\zeta) .$$

6.5. Associated Polynomials for D_n . In the case of D_n we have

$$q_n(x) = 4xT_{n-1}(x) .$$

Therefore,

$$a_n(x) = 2xT_{n-1}\left(\frac{x}{2}\right) ,$$

and

$$Q_n(x) = 2x^n \left(x + \frac{1}{x}\right) T_{n-1}\left(\frac{1}{2}\left(x + \frac{1}{x}\right)\right) .$$

Set $x = e^{i\theta}$ to obtain

$$\begin{aligned}
Q_n(x) &= 2(x^{n+1} + x^{n-1})T_{n-1}(\cos \theta) \\
&= 2(x^{n+1} + x^{n-1})\cos(n-1)\theta \\
&= 2(x^{n+1} + x^{n-1})\frac{1}{2}\left(e^{i(n-1)\theta} + e^{-i(n-1)\theta}\right) \\
&= x^{2n} + x^{2(n-1)} + x^2 + 1.
\end{aligned}$$

Therefore, $Q_n(x) = x^{2n} + x^{2(n-1)} + x^2 + 1$ for all $x \in \mathbf{C}$. We conclude that $f_n(x) = x^n + x^{n-1} + x + 1$. We present the formula for $a_n(x)$ and the factorization of $f_n(x)$ for small values of n .

$$\begin{aligned}
\bullet D_4 \quad a_4 &= x^4 - 3x^2 & f_4(x) &= \Phi_2^2 \Phi_6 \\
\bullet D_5 \quad a_5 &= x^5 - 4x^3 + 2x & f_5 &= \Phi_2 \Phi_8 \\
\bullet D_6 \quad a_6 &= x^6 - 5x^4 + 5x^2 & f_6 &= \Phi_2^2 \Phi_{10} \\
\bullet D_7 \quad a_7 &= x^7 - 6x^5 + 9x^3 - 2x & f_7 &= \Phi_2 \Phi_4 \Phi_{12} \\
\bullet D_8 \quad a_8 &= x^8 - 7x^6 + 14x^4 - 7x^2 & f_8 &= \Phi_2^2 \Phi_{14} \\
\bullet D_9 \quad a_9 &= x^9 - 8x^7 + 20x^5 - 16x^3 + 2x & f_9 &= \Phi_2 \Phi_{16} \\
\bullet D_{10} \quad a_{10} &= x^{10} - 9x^8 + 27x^6 - 30x^4 + 9x^2 & f_{10} &= \Phi_2^2 \Phi_6 \Phi_{18}.
\end{aligned}$$

Write $n-1 = 2^\alpha N$ where N is odd and $r = 2^{\alpha+1}$.

$$f_n(x) = (x+1)(x^{n-1} + 1) = \Phi_2(x) \prod_{d|N} \Phi_{rd}(x).$$

Proposition 8.

$$(24) \quad Q_n(x) = \Phi_4(x) \prod_{\substack{d|n-1 \\ d \text{ odd}}} \Phi_{2rd}(x).$$

7. FACTORIZATION OF CHEBYSHEV POLYNOMIALS

7.1. Chebyshev polynomials of second kind. It is well-known that the roots of U_n are given by

$$x_k = \cos \frac{k\pi}{n+1} \quad k = 1, 2, \dots, n,$$

as we already observed in (11).

We can write them in the form $x_k = \cos k\theta$, where $\theta = \frac{\pi}{n+1}$.

The roots of $a_n(x) = U_n(\frac{x}{2})$ are

$$\lambda_k = 2 \cos \frac{k\pi}{n+1} = 2 \cos k\theta \quad k = 1, 2, \dots, n,$$

i.e. the roots of $a_n(x)$ are

$$2 \cos \frac{m_i \pi}{h}$$

where m_i are the exponents of A_n and h is the Coxeter number for A_n .

The roots of $a_n(x) = U_n(\frac{x}{2})$ are

$$\lambda_k = 2 \cos \frac{k\pi}{h} = 2 \cos k\theta \quad k = 1, 2, \dots, n.$$

Denote them by

$$\lambda_1 = 2 \cos \theta_1 = 2 \cos \theta, \quad \lambda_2 = 2 \cos \theta_2, \dots, \quad \lambda_n = 2 \cos \theta_n.$$

Note that $\theta_k = k\theta$ and

$$\theta_j + \theta_{n+1-j} = \pi.$$

This implies that $\lambda_j = -\lambda_{n-j+1}$. As a result

$$(25) \quad \{\lambda_j | j = 1, 2, \dots, n\} = \{-\lambda_j | j = 1, 2, \dots, n\}.$$

The roots of $p_n(x)$ are then

$$\xi_k = 2 + \lambda_k = 2 + 2 \cos \frac{k\pi}{n+1} = 2 + 2 \cos \theta_k = 4 \cos^2 \frac{\theta_k}{2}.$$

It follows from (25) that the eigenvalues of C occur in pairs $\{2 + \lambda, 2 - \lambda\}$. This is a general result which holds for each Cartan matrix corresponding to a simple Lie algebra over \mathbf{C} , see [2, p. 345] for a general proof.

It follows from Theorem 5 that the roots of $Q_n(x)$ are

$$e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n}, e^{-i\theta_1}, e^{-i\theta_2}, \dots, e^{-i\theta_n},$$

or equivalently,

$$e^{i\theta}, e^{2i\theta}, \dots, e^{ni\theta}, e^{(n+2)i\theta}, e^{(n+3)i\theta}, \dots, e^{(2n+1)i\theta}.$$

Note that

$$e^{i\theta} = e^{\frac{i\pi}{h}} = e^{\frac{2\pi i}{2h}}.$$

As a result $e^{i\theta}$ is a $(2h)$ th primitive root of unity and therefore a root of the cyclotomic polynomial Φ_{2h} . The other roots of this cyclotomic polynomial are of course $e^{ki\theta}$ where $(k, 2h) = 1$. This determines Φ_{2h} as an irreducible factor of $Q_n(x)$. To determine the other irreducible factors we proceed as follows: The root

$$e^{2i\theta} = e^{\frac{2\pi i}{h}}$$

is a primitive h th root of unity and the other roots of Φ_h are $e^{ki\theta}$ where $(k, h) = 1$. In general for each d which is a divisor of $2h$ (but $d \neq 1, 2$) we form Φ_d by grouping together all the $e^{ki\theta}$ such that $(k, 2h) = \frac{2h}{d}$. Since $e^{(n+1)i\theta}$ and $e^{2ni\theta}$ do not appear as roots of $Q_n(x)$ the cyclotomic polynomials Φ_1 and Φ_2 do not appear in the factorization. You can consider this argument as another proof of Proposition 22.

The roots of $a_n(x)$ are of the form $e^{i\theta_k} + e^{-i\theta_k}$, where $e^{i\theta_k}$ is a root of $Q_n(x)$. It is easy to see that each irreducible factor of $Q_n(x)$ determines an irreducible factor of $a_n(x)$ and conversely. In fact this is the argument of Lehmer in [18]. If Φ_d is a cyclotomic factor of $Q_n(x)$ then Φ_d being a reciprocal polynomial it can be written in the form

$$\Phi_d(x) = x^m \psi_d(x + \frac{1}{x})$$

where $m = \deg \psi_n = \frac{1}{2}\phi(d)$. According to Lehmer the polynomial ψ_d is irreducible. These polynomials are all the irreducible factors of $a_n(x)$. This fact is easily established by looking at the roots of $a_n(x)$. The way to determine the irreducible factors of $a_n(x)$ is the following: Start with a cyclotomic factor of $Q_n(x)$. For example, consider $\Phi_{2h}(x)$. The roots of this polynomial are $e^{ki\theta}$ where $(k, 2h) = 1$. Take only $e^{ki\theta}$ with $(k, 2h) = 1$ such that $1 \leq k \leq n$. The corresponding roots of $a_n(x)$ are of course $2 \cos k\theta$, $1 \leq k \leq n$ and $(k, 2h) = 1$. This determines the polynomial ψ_{2h} . Then we repeat this procedure with the other cyclotomic factors.

Example 5. To determine the factorization of $U_5(x) = 32x^5 - 32x^3 + 6x$. Since $n = 5$, $h = n + 1 = 6$ and $\theta = \frac{\pi}{6}$. The roots of U_5 are

$$\cos \frac{\pi}{6} \quad \cos \frac{2\pi}{6} \quad \cos \frac{3\pi}{6} \quad \cos \frac{4\pi}{6} \quad \cos \frac{5\pi}{6}.$$

The roots of $a_5(x)$ are

$$\lambda_1 = 2 \cos \frac{\pi}{6} \quad \lambda_2 = 2 \cos \frac{2\pi}{6} \quad \lambda_3 = 2 \cos \frac{3\pi}{6} \quad \lambda_4 = 2 \cos \frac{4\pi}{6} \quad \lambda_5 = 2 \cos \frac{5\pi}{6}.$$

The roots of $Q_5(x)$ are

$$e^{\frac{\pi}{6}}, e^{\frac{2\pi}{6}}, e^{\frac{3\pi}{6}}, e^{\frac{4\pi}{6}}, e^{\frac{5\pi}{6}}, e^{\frac{7\pi}{6}}, e^{\frac{8\pi}{6}}, e^{\frac{9\pi}{6}}, e^{\frac{10\pi}{6}}, e^{\frac{11\pi}{6}}.$$

We group together all $e^{ik\theta}$ such that $(k, 12) = 1$ i.e.,

$$e^{\frac{\pi}{6}}, e^{\frac{5\pi}{6}}, e^{\frac{7\pi}{6}}, e^{\frac{11\pi}{6}}$$

which are the roots of $\Phi_{12}(x)$. Note that

$$e^{\frac{\pi}{6}} + e^{\frac{11\pi}{6}} = e^{\frac{\pi}{6}} + e^{-\frac{\pi}{6}} = 2 \cos \frac{\pi}{6} = \lambda_1 = \sqrt{3} ,$$

and

$$e^{\frac{5\pi}{6}} + e^{\frac{7\pi}{6}} = e^{\frac{5\pi}{6}} + e^{-\frac{5\pi}{6}} = \cos \frac{5\pi}{6} = \lambda_5 = -\sqrt{3} .$$

These roots λ_1 and λ_5 are roots of $\psi_{12} = x^2 - 3$ which is an irreducible factor of $a_5(x)$.

Then we group together all $e^{ik\theta}$ such that $(k, 12) = 2$ i.e.,

$$e^{\frac{2\pi}{6}}, e^{\frac{10\pi}{6}}$$

which are the roots of $\Phi_6(x)$. Note that

$$e^{\frac{2\pi}{6}} + e^{\frac{10\pi}{6}} = e^{\frac{2\pi}{6}} + e^{-\frac{2\pi}{6}} = 2 \cos \frac{2\pi}{6} = \lambda_2 = 1 .$$

Therefore $x - 1$ is the irreducible factor of $a_5(x)$ corresponding to $\Phi_6(x)$.

Then we group together all $e^{ik\theta}$ such that $(k, 12) = 3$ i.e.,

$$e^{\frac{3\pi}{6}}, e^{\frac{9\pi}{6}}$$

which are the roots of $\Phi_4(x)$. Note that

$$e^{\frac{3\pi}{6}} + e^{\frac{9\pi}{6}} = e^{\frac{3\pi}{6}} + e^{-\frac{3\pi}{6}} = 2 \cos \frac{3\pi}{6} = \lambda_3 = 0 .$$

Therefore $\psi_4(x) = x$ is the irreducible factor of $a_5(x)$ corresponding to $\Phi_4(x)$.

Finally we group together all $e^{ik\theta}$ such that $(k, 12) = 4$ i.e.,

$$e^{\frac{4\pi}{6}}, e^{\frac{8\pi}{6}}$$

which are the roots of $\Phi_3(x)$. Note that

$$e^{\frac{4\pi}{6}} + e^{\frac{8\pi}{6}} = e^{\frac{4\pi}{6}} + e^{-\frac{4\pi}{6}} = 2 \cos \frac{4\pi}{6} = \lambda_4 = -1 .$$

Therefore $\psi_3(x) = x + 1$ is the irreducible factor of $a_5(x)$ corresponding to $\Phi_3(x)$.

We end up with the integer factorization of $a_5(x)$ into irreducible factors:

$$a_5(x) = x(x+1)(x-1)(x^2-3) .$$

Since $U_5(x) = a_5(2x)$ we obtain the factorization of $U_5(x)$:

$$U_5(x) = 2x(2x+1)(2x-1)(4x^2-3) .$$

In the following list we give the polynomial ψ_n corresponding to each cyclotomic polynomial Φ_n up to $n = 24$.

$$\begin{aligned}
\psi_3(x) &= x + 1 \\
\psi_4(x) &= x \\
\psi_5(x) &= x^2 + x - 1 \\
\psi_6(x) &= x - 1 \\
\psi_7(x) &= x^3 + x^2 - 2x - 1 \\
\psi_8(x) &= x^2 - 2 \\
\psi_9(x) &= x^3 - 3x + 1 \\
\psi_{10}(x) &= x^2 - x - 1 \\
\psi_{11}(x) &= x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1 \\
\psi_{12}(x) &= x^2 - 3 \\
\psi_{13}(x) &= x^6 + x^5 - 5x^4 - 4x^3 + 6x^2 + 3x - 1 \\
\psi_{14}(x) &= x^3 - x^2 - 2x + 1 \\
\psi_{15}(x) &= x^4 - x^3 - 4x^2 + 4x + 1 \\
\psi_{16}(x) &= x^4 - 4x^2 + 2 \\
\psi_{17}(x) &= x^8 + x^7 - 7x^6 - 6x^5 + 15x^4 + 10x^3 - 10x^2 - 4x + 1 \\
\psi_{18}(x) &= x^3 - 3x - 1 \\
\psi_{19}(x) &= x^9 + x^8 - 8x^7 - 7x^6 + 21x^5 + 15x^4 - 20x^3 - 10x^2 + 5x + 1 \\
\psi_{20}(x) &= x^4 - 5x^2 + 5 \\
\psi_{21}(x) &= x^6 - x^5 - 6x^4 + 6x^3 + 8x^2 - 8x + 1 \\
\psi_{22}(x) &= x^5 - x^4 - 4x^3 + 3x^2 + 3x - 1 \\
\psi_{23}(x) &= x^{11} + x^{10} - 10x^9 - 9x^8 + 36x^7 + 28x^6 - 56x^5 - 35x^4 + 35x^3 + 15x^2 - 6x - 1 \\
\psi_{24}(x) &= x^4 - 4x^2 + 1 .
\end{aligned}$$

Remark 5. Since $\psi_n(x)$ is the minimal polynomial of $2 \cos \frac{2\pi}{n}$ it is reasonable to define $\psi_1(x) = x - 2$ and $\psi_2(x) = x + 2$. They correspond to the reducible $Q_1(x) = (x - 1)^2$ and $Q_2(x) = (x + 1)^2$ respectively.

Remark 6. To compute ψ_n is straightforward. We give two examples.

- $n = 36$. Since

$$\Phi_{36} = x^{12} - x^6 + 1$$

the polynomial ψ_{36} is of degree 6. We need

$$x^{12} - x^6 + 1 = x^6 \psi_{36}(\zeta)$$

where $\zeta = x + \frac{1}{x}$. Therefore

$$\psi_{36}(\zeta) = x^6 + \frac{1}{x^6} - 1 = (\zeta^6 - 6\zeta^4 + 9\zeta^2 - 2) - 1 = \zeta^6 - 6\zeta^4 + 9\zeta^2 - 3 .$$

- $n = 60$. Since

$$\Phi_{60} = x^{16} + x^{14} - x^{10} - x^8 - x^6 + x^2 + 1$$

the polynomial ψ_{60} is of degree 8. We need

$$x^{16} + x^{14} - x^{10} - x^8 - x^6 + x^2 + 1 = x^8 \psi_{60}(\zeta)$$

where $\zeta = x + \frac{1}{x}$. Therefore

$$\begin{aligned}
\psi_{60}(\zeta) &= (x^8 + \frac{1}{x^8}) + (x^6 + \frac{1}{x^6}) - (x^2 + \frac{1}{x^2}) - 1 \\
&= a_8(\zeta) + a_6(\zeta) - a_2(\zeta) - 1 \\
&= \zeta^8 - 7\zeta^6 + 14\zeta^4 - 8\zeta^2 + 1 .
\end{aligned}$$

Example 6. To find the factorization of

$$U_9(x) = 512x^9 - 1024x^7 + 672x^5 - 160x^3 + 10x .$$

Since

$$Q_9(x) = \Phi_4\Phi_5\Phi_{10}\Phi_{20} ,$$

we have that

$$a_9(x) = \psi_4\psi_5\psi_{10}\psi_{20} = x(x^2 + x - 1)(x^2 - x - 1)(x^4 - 5x^2 + 5) .$$

Finally

$$U_9(x) = a_9(2x) = 2x(4x^2 + 2x - 1)(4x^2 - 2x - 1)(16x^4 - 20x^2 + 5) .$$

To conclude we state the following result:

Proposition 9.

$$U_n(x) = \prod_{\substack{j|2n+2 \\ j \neq 1,2}} \psi_j(2x) .$$

7.2. Chebyshev polynomials of the first kind. The roots of T_n are given by

$$x_k = \cos \frac{(2k-1)\pi}{2n} \quad k = 1, 2, \dots, n .$$

Let $h = 2n$ (h is the Coxeter number of the root system B_n) and $\theta = \frac{\pi}{h}$. Then

$$x_k = \cos(2k-1)\theta \quad k = 1, 2, \dots, n ,$$

i.e. the roots of T_n are

$$\cos k\theta$$

where k runs over the exponents of a root system of type B_n .

The roots of $a_n(x) = 2T_n(\frac{x}{2})$ are

$$\lambda_k = 2 \cos \frac{(2k-1)\pi}{h} = 2 \cos(2k-1)\theta \quad k = 1, 2, \dots, n .$$

Denote them by

$$\lambda_1 = 2 \cos \theta_1 = 2 \cos \theta, \quad \lambda_2 = 2 \cos \theta_2, \dots, \quad \lambda_n = 2 \cos \theta_n .$$

Note that $\theta_k = (2k-1)\theta$ and

$$\theta_j + \theta_{n+1-j} = \pi .$$

This implies that $\lambda_j = -\lambda_{n-j+1}$. As a result

$$(26) \quad \{\lambda_j | j = 1, 2, \dots, n\} = \{-\lambda_j | j = 1, 2, \dots, n\} .$$

The roots of $p_n(x)$ are then

$$\xi_k = 2 + \lambda_k = 2 + 2 \cos \frac{(2k-1)\pi}{2n} = 2 + 2 \cos \theta_k = 4 \cos^2 \frac{\theta_k}{2} .$$

It follows from (26) that the eigenvalues of C occur in pairs $\{2 + \lambda, 2 - \lambda\}$.

It follows from Theorem 5 that the roots of $Q_n(x)$ are

$$e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n}, e^{-i\theta_1}, e^{-i\theta_2}, \dots, e^{-i\theta_n} ,$$

or equivalently,

$$e^{i\theta}, e^{3i\theta}, e^{5i\theta}, \dots, e^{(2n-1)i\theta}, e^{(2n+1)i\theta}, e^{(2n+3)i\theta}, \dots, e^{(4n-1)i\theta} .$$

Note that

$$e^{i\theta} = e^{\frac{i\pi}{h}} = e^{\frac{2\pi i}{2h}} = e^{\frac{2\pi i}{4n}} .$$

As a result $e^{i\theta}$ is a $(2h)$ th primitive root of unity and therefore a root of the cyclotomic polynomial $\Phi_{2h} = \Phi_{4n}$. The other roots of this cyclotomic polynomial are of course $e^{ki\theta}$ where $(k, 2h) = 1$ and k odd. This determines Φ_{2h} as an irreducible

factor of $Q_n(x)$. To determine the other irreducible factors we proceed as follows: Take an odd divisor d of $4n$. It is of course an odd divisor of n as well. If we write $n = 2^\alpha N$ where N is odd, this divisor d is also a divisor of N . Use the notation $r = 2^{\alpha+2}$ and note that $2h = 4n = rN$. We group together all $e^{ik\theta}$ where k is odd and $(N, k) = \frac{N}{d}$. Note that $d = N$ corresponds to Φ_{4n} . These roots define Φ_{rd} . You can consider this argument as another proof of Proposition 23.

As in the case of A_n the roots of $a_n(x)$ are of the form $e^{i\theta_k} + e^{-i\theta_k}$, where $e^{i\theta_k}$ is a root of $Q_n(x)$. Again, the irreducible factor of $Q_n(x)$ are in one-to-one correspondence with the irreducible factors of $a_n(x)$. We denote the irreducible factors of $a_n(x)$ with ψ_n as before.

Example 7. To determine the factorization of

$$T_5(x) = T_5(x) = 16x^5 - 20x^3 + 5x .$$

Since $n = 5$, $h = 2n = 10$ and $\theta = \frac{\pi}{10}$. The roots of T_5 are

$$\cos \frac{\pi}{10} \quad \cos \frac{3\pi}{10} \quad \cos \frac{5\pi}{10} \quad \cos \frac{7\pi}{10} \quad \cos \frac{9\pi}{10} .$$

The roots of $a_5(x)$ are

$$\lambda_1 = 2 \cos \frac{\pi}{10} \quad \lambda_2 = 2 \cos \frac{3\pi}{10} \quad \lambda_3 = 2 \cos \frac{5\pi}{10} \quad \lambda_4 = 2 \cos \frac{7\pi}{10} \quad \lambda_5 = 2 \cos \frac{9\pi}{10} .$$

Therefore, the roots of $Q_5(x)$ are

$$e^{\frac{\pi}{10}}, e^{\frac{3\pi}{10}}, e^{\frac{5\pi}{10}}, e^{\frac{7\pi}{10}}, e^{\frac{9\pi}{10}}, e^{\frac{11\pi}{10}}, e^{\frac{13\pi}{10}}, e^{\frac{15\pi}{10}}, e^{\frac{17\pi}{10}}, e^{\frac{19\pi}{10}} .$$

We group together

$$e^{\frac{\pi}{10}}, e^{\frac{3\pi}{10}}, e^{\frac{7\pi}{10}}, e^{\frac{9\pi}{10}}, e^{\frac{11\pi}{10}}, e^{\frac{13\pi}{10}}, e^{\frac{17\pi}{10}}, e^{\frac{19\pi}{10}}$$

which are the roots of $\Phi_{20}(x)$. These are all the exponentials $e^{ik\theta}$ with k odd and $(k, 20) = 1$.

Note that

$$e^{\frac{\pi}{10}} + e^{\frac{19\pi}{10}} = e^{\frac{\pi}{10}} + e^{-\frac{\pi}{10}} = 2 \cos \frac{\pi}{10} = \lambda_1 ,$$

$$e^{\frac{3\pi}{10}} + e^{\frac{17\pi}{10}} = e^{\frac{3\pi}{10}} + e^{-\frac{3\pi}{10}} = 2 \cos \frac{3\pi}{10} = \lambda_2 ,$$

$$e^{\frac{7\pi}{10}} + e^{\frac{13\pi}{10}} = e^{\frac{7\pi}{10}} + e^{-\frac{7\pi}{10}} = 2 \cos \frac{7\pi}{10} = \lambda_4 ,$$

$$e^{\frac{9\pi}{10}} + e^{\frac{11\pi}{10}} = e^{\frac{9\pi}{10}} + e^{-\frac{9\pi}{10}} = 2 \cos \frac{9\pi}{10} = \lambda_5 ,$$

These roots λ_1 , λ_2 , λ_4 and λ_5 are roots of $\psi_{20} = x^4 - 5x^2 + 5$ which is an irreducible factor of $a_5(x)$.

The only other odd divisor of 20 is 5. Therefore we group together

$$e^{\frac{5\pi}{10}}, e^{\frac{15\pi}{10}}$$

which are the roots of $\Phi_4(x)$. These are all the exponentials $e^{ik\theta}$ with k odd and $(k, 20) = 5$. Noting that $20 = 2^2 \cdot 5$ we see that 1, 5 are just the positive divisors of 5.

Note that

$$e^{\frac{5\pi}{10}} + e^{\frac{15\pi}{10}} = e^{\frac{5\pi}{10}} + e^{-\frac{5\pi}{10}} = 2 \cos \frac{5\pi}{10} = \lambda_3 = 0 .$$

Therefore x is the irreducible factor of $a_5(x)$ corresponding to $\Phi_4(x)$.

We end up with the integer factorization of $a_5(x)$ into irreducible factors:

$$a_5(x) = x(x^4 - 5x^2 + 5) .$$

TABLE 4. Roots of polynomials in terms of $\theta_j = \frac{\pi m_j}{h}$.

Polynomial	Roots
q_n	$\cos \theta_j$
a_n	$2 \cos \theta_j$
p_n	$4 \cos^2 \frac{\theta_j}{2}$
Q_n	$e^{i\theta_j}$
f_n	$e^{2i\theta_j}$

Since $T_5(x) = \frac{1}{2}a_5(2x)$ we obtain the factorization of $T_5(x)$:

$$T_5(x) = x(16x^4 - 20x^2 + 5) .$$

Example 8. To find the factorization of

$$T_9(x) = 256x^9 - 576x^7 + 432x^5 + 20x^3 + 9x .$$

Since

$$Q_9(x) = \Phi_4 \Phi_{12} \Phi_{36} ,$$

we have that

$$a_9(x) = \psi_4 \psi_{12} \psi_{36} .$$

Since $\psi_4(x) = x$ and $\psi_{12}(x) = x^2 - 3$ and $\psi_{36} = x^6 - 6x^4 + 9x^2 - 2$ (Remark 6) we have

$$a_9(x) = x(x^2 - 3)(x^6 - 6x^4 + 9x^2 - 3) .$$

Finally

$$T_9(x) = \frac{1}{2}a_9(2x) = x(4x^2 - 3)(64x^6 - 96x^4 + 36x^2 - 3) .$$

To conclude we state the following result:

Proposition 10. Let $n = 2^\alpha N$ where N is odd and let $r = 2^{\alpha+2}$. Then

$$T_n(x) = \frac{1}{2} \prod_{j|N} \psi_{rj}(2x) .$$

Remark 7. In the case of D_n

$$q_n(x) = 4xT_{n-1}(x) .$$

Therefore

$$a_n(x) = 2xT_{n-1}(x) .$$

$$\begin{aligned} a_n(x_0) = 0 & \quad \Rightarrow \quad 2x_0T_{n-1}(x_0) = 0 \\ \Rightarrow \quad x_0 = 0, \quad \text{or} \quad x_0 = 2 \cos \frac{(2k-1)\pi}{2(n-1)} \quad k = 1, 2, \dots, n-1 . \end{aligned}$$

In summary: The roots of $a_n(x)$ are

$$2 \cos \frac{m_i \pi}{h}$$

where h is the Coxeter number for D_n and m_i are the exponents. It follows that 0 is always a root and $a_n(x) = xg_n(x)$ where $g_n(x)$ is the a_{n-1} characteristic polynomial for B_{n-1} .

In table 4 we list the roots of the various polynomials that we considered.

8. EXCEPTIONAL LIE ALGEBRAS

8.1. G_2 **graphs.** The Cartan matrix for G_2 is

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

We can define a generalization of the Cartan matrix of G_2 by defining the matrix G_n to be the same as the matrix of A_n except at the $(n, n-1)$ position we replace -1 with -3 . Then $\det G_n = 3 - n$ and we find that

$$q_n(x) = 2T_n(x) - U_{n-2}(x).$$

The characteristic polynomial for a Lie algebra of type G_2 is

$$p_2(x) = x^2 - 4x + 1,$$

since

$$q_2(x) = 2T_2(x) - U_0(x) = 4x^2 - 3$$

and

$$p_2(x) = q_2\left(\frac{x}{2} - 1\right) = x^2 - 4x + 1.$$

The roots of $a_2(x) = x^2 - 3$ are

$$2 \cos \frac{m_i \pi}{h}$$

where $m_1 = 1$ and $m_2 = 5$ are the exponents of root system of type G_2 . The Coxeter number h is 6.

Finally,

$$Q_2(x) = x^4 - x^2 + 1 = \Phi_{12}(x),$$

and

$$f_2(x) = x^2 - x + 1 = \Phi_6(x).$$

Note that 1, 5 are the positive integers less than six and relatively prime to 6. This explains the appearance of Φ_6 .

For $n = 3$ we have the affine Lie algebra $g_2^{(1)}$. We record the following formulas:

$$q_3(x) = 8x^3 - 8x$$

$$p_3(x) = x^3 - 6x^2 + 8x$$

$$a_3(x) = x^3 - 4x = x(x^2 - 4) = \psi_1(x)\psi_2(x)\psi_4(x)$$

$$Q_3(x) = x^6 - x^4 - x^2 + 1 = \Phi_1^2 \Phi_2^2 \Phi_4$$

$$f_3(x) = x^3 - x^2 - x + 1 = \Phi_1^2 \Phi_2.$$

The spectrum of the graph is $-2, 0, 2$.

For $n = 4$ we have the following formulas:

$$q_4(x) = 16x^4 - 20x^2 + 3$$

$$p_4(x) = x^4 - 8x^3 + 19x^2 - 12x - 1$$

$$a_4(x) = x^4 - 5x^2 + 3$$

$$Q_4(x) = x^8 - x^6 - x^4 - x^2 + 1$$

$$f_4(x) = x^4 - x^3 - x^2 - x + 1.$$

Note that the Coxeter polynomial of the graph G_4 is no longer cyclotomic. It has Malher measure approximately 1.72208.

8.2. **Graph of type F_4 .** The Cartan matrix for F_4 is

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

$$p_4(x) = x^4 - 8x^3 + 20x^2 - 16x + 1,$$

and

$$a_4(x) = x^4 - 4x^2 + 1 = \psi_{24}(x).$$

The roots of $a_4(x)$ are

$$\frac{1}{2}(\pm\sqrt{6} \pm \sqrt{2})$$

i.e.

$$2 \cos \frac{m_i \pi}{12}$$

where $m_i \in \{1, 5, 7, 11\}$. These are the exponents for F_4 and being the numbers less than 12 and prime to 12 imply

$$f_4(x) = x^4 - x^2 + 1 = \Phi_{12}(x).$$

8.3. **E_n graphs.** One can defined a generalized E_n diagram of the same form as A_n except that $a_{21} = a_{12} = a_{23} = a_{32} = 0$ and $a_{13} = a_{31} = a_{24} = a_{42} = -1$. It turns out that $q_n(x)$ is equal to $2x$ times the q_n of D_{n-1} minus the q_n of A_{n-2} . We therefore obtain

$$q_n(x) = (2x) \cdot 4x T_{n-2}(x) - U_{n-2}(x) = 8x^2 T_{n-2}(x) - U_{n-2}(x).$$

- $n = 6$

The Cartan matrix for E_6 is

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

$$q_6(x) = 64x^6 - 80x^4 + 20x^2 - 1 = (2x+1)(2x-1)(16x^4 - 16x^2 + 1)$$

$$p_6(x) = (x-1)(x-3)(x^4 - 8x^3 + 20x^2 - 16x + 1)$$

$$a_6(x) = x^6 - 5x^4 + 5x^2 - 1 = (x+1)(x-1)(x^4 - 4x^2 + 1) = \psi_3(x)\psi_6(x)\psi_{24}(x)$$

$$Q_6(x) = (x^2 + x + 1)(x^2 - x + 1)(x^8 - x^4 + 1) = \Phi_3(x)\Phi_6(x)\Phi_{24}(x).$$

The exponents of E_6 are $\{1, 4, 5, 7, 8, 11\}$ and the Coxeter number is 12. The subset $\{1, 5, 7, 11\}$ produces Φ_{12} and $\{4, 8\}$ produces Φ_3 . Therefore

$$f_6(x) = \Phi_3(x)\Phi_{12}(x).$$

The roots of $a_6(x)$ are

$$\pm 1, \frac{1}{2}(\pm\sqrt{6} \pm \sqrt{2})$$

i.e.

$$2 \cos \frac{m_i \pi}{12}$$

where $m_i \in \{1, 4, 5, 7, 8, 11\}$. These are the exponents for E_6 . The Coxeter number is 12.

- $n = 7$

The Cartan matrix for E_7 is

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

$$q_7(x) = 128x^7 - 192x^5 + 72x^3 - 6x = 2x(64x^6 - 96x^4 + 36x^2 - 3)$$

$$p_7(x) = (x-2)(x^6 - 12x^5 + 54x^4 - 112x^3 + 105x^2 + 1)$$

$$a_7(x) = x^7 - 6x^5 + 9x^3 - 3x = x(x^6 - 6x^4 + 9x^2 - 3) = \psi_4(x)\psi_{36}(x)$$

$$Q_7(x) = (x^2 + 1)(x^{12} - x^6 + 1) = \Phi_4(x)\Phi_{36}(x).$$

The exponents of E_7 are $\{1, 5, 7, 9, 11, 13, 17\}$ and the Coxeter number is 18. The subset $\{1, 5, 7, 11, 13, 17\}$ produces Φ_{18} and $\{9\}$ produces Φ_2 . Therefore

$$f_7(x) = \Phi_2(x)\Phi_{18}(x).$$

- $n = 8$

The Cartan matrix for E_8 is

$$\begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}.$$

$$q_8(x) = 256x^8 - 448x^6 + 224x^4 - 32x^2 + 1$$

$$p_8(x) = x^8 - 16x^7 + 105x^6 + 364x^5 + 714x^4 - 784x^3 + 440x^2 - 96x + 1$$

$$a_8(x) = x^8 - 7x^6 + 14x^4 - 8x^2 + 1 = \psi_{60}(x)$$

$$Q_8(x) = x^{16} + x^{14} - x^{10} - x^8 - x^6 + x^2 + 1 = \Phi_{60}(x).$$

The exponents of E_8 are $\{1, 7, 11, 13, 17, 19, 23, 29\}$ which are the positive integers less than 30 and prime to 30. Therefore

$$f_8(x) = \Phi_{30}(x).$$

Since

$$p_n(x) = q_n\left(\frac{x}{2} - 1\right)$$

we have that

$$p_n(0) = q_n(-1) = (-1)^{n-2} (8T_{n-2}(1) - U_{n-2}(1)).$$

Therefore the determinant of a diagram of type E_n is $8T_{n-2}(1) - U_{n-2}(1) = 9 - n$. This gives determinants 3, 2, 1 respectively for E_6 , E_7 , E_8 .

Example 9. Let us consider $n = 9$. Of course E_9 is the same as $E_8^{(1)}$ the affine E_8 diagram. In this case

$$q_9(x) = 512x^9 - 1024x^7 + 640x^5 - 136x^3 + 8x$$

and

$$a_9(x) = x(x-1)(x+1)(x+2)(x-2)(x^2-x-1)(x^2+x-1) = \psi_1(x)\psi_2(x)\psi_3(x)\psi_4(x)\psi_5(x)\psi_{10}(x) .$$

$a_9(x)$ is the characteristic polynomial of the adjacency matrix, so the spectrum of the $E_8^{(1)}$ graph is

$$0, 1, -1, 2, -2, \tau, \frac{1}{\tau}, -\tau, -\frac{1}{\tau}$$

where $\tau = \frac{1+\sqrt{5}}{2} = 2 \cos \frac{\pi}{5}$.

For the reader who is familiar with the Mackay correspondence, these are the values in the character table of the binary icosahedral group $SL(2, 5)$.

Note that

$$Q_9 = \Phi_1^2 \Phi_2^2 \Phi_3 \Phi_4 \Phi_5 \Phi_6 \Phi_{10}$$

and

$$f_9(x) = x^9 + x^8 - x^6 - x^5 - x^4 - x^3 + x + 1 = \Phi_1^2(x)\Phi_2(x)\Phi_3(x)\Phi_5(x) .$$

Example 10. Let us consider E_{10} . In this case we obtain

$$q_{10}(x) = 1024x^{10} - 2304x^8 + 1728x^6 - 496x^4 + 48x^2 - 1$$

$$a_{10}(x) = x^{10} - 9x^8 + 27x^6 - 31x^4 + 12x^2 - 1$$

$$Q_{10}(x) = x^{20} + x^{18} - x^{14} - x^{12} - x^{10} - x^8 - x^6 + x^2 + 1$$

$$f_{10}(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1 .$$

We recognize f_{10} as the famous Lehmer polynomial (2). Its largest real root is about 1.17628 which is the smallest known Salem number.

9. THE SINE FORMULA

To prove the sine formula we only will need the following Lemma which was already proved by case to case verification:

Lemma 2. The roots of $a_n(x)$ are

$$2 \cos \frac{m_i \pi}{h}$$

where m_i are the exponents of \mathfrak{g} and h is the Coxeter number of \mathfrak{g} .

We give some references and history on how to prove this Lemma without a case by case verification: There is an empirical procedure due to H.M. Coxeter which can be used to find the roots of the Coxeter polynomial. If ζ is a primitive h root of unity (where h is the Coxeter number) then the roots of the Coxeter polynomial are ζ^m where m runs over the exponents of the corresponding root system [3], [6]. This observation allows the calculation of the Coxeter polynomial for each root system. It also explains the duality (20) since non-real eigenvalues of R appear in conjugate pairs. Coxeter also observed that

$$(27) \quad h\ell = 2r$$

where r is the number of positive roots. Using (27) as the only empirical fact Coleman proved in [3] the procedure of Coxeter. The proof of (27) is in the classic paper of Kostant of 1958 [17]. Knowing the roots of the Coxeter polynomial, it is straightforward to determine the spectrum of the Cartan matrix C , see e.g. [2, Theorem 2]. This in turns determines the roots of $a_n(x)$ via the relation (1).

We have seen in Lemma (2) that the roots of $a_n(x)$ are

$$2 \cos \frac{m_i \pi}{h}$$

where m_i are the exponents of \mathfrak{g} and h is the Coxeter number of \mathfrak{g} . Let $\theta_i = \frac{m_i \pi}{h}$. Then the roots of $a_n(x)$ are $\lambda_i = 2 \cos \theta_i$, $i = 1, 2, \dots, \ell$.

Recall the duality property of the exponents (20).

$$m_i + m_{\ell+1-i} = h .$$

It follows that

$$m_i \frac{\pi}{h} + m_{\ell+1-i} \frac{\pi}{h} = \pi .$$

As a result:

$$\theta_i + \theta_{\ell+1-i} = \pi .$$

Using this formula, we can infer a relationship satisfied by the roots of $p_n(x)$ which are:

$$\xi_i = 4 \cos^2 \frac{\theta_i}{2} .$$

Namely:

$$\begin{aligned} \xi_i + \xi_{\ell+1-i} &= 4 \cos^2 \frac{\theta_i}{2} + 4 \cos^2 \frac{\theta_{\ell+1-i}}{2} \\ &= 4 \left(\cos^2 \frac{\theta_i}{2} + \cos^2 \frac{\pi - \theta_i}{2} \right) \\ &= 4 \left(\cos^2 \frac{\theta_i}{2} + \sin^2 \frac{\theta_i}{2} \right) \\ &= 4 \end{aligned}$$

It follows that the eigenvalues of the Cartan matrix occur in pairs $\xi, 4 - \xi$.

Remark 8. *The fact that the eigenvalues of C occur in pairs $\{\xi, 4 - \xi\}$, and a different line of proof can be found in [2, p. 345].*

Theorem 3. *Let \mathfrak{g} be a complex simple Lie algebra of rank ℓ , h the Coxeter number, m_1, m_2, \dots, m_ℓ the exponents of \mathfrak{g} and C the Cartan matrix. Then*

$$2^{2\ell} \prod_{i=1}^{\ell} \sin^2 \frac{m_i \pi}{2h} = \det C .$$

Proof. It follows from Lemma (2) that

$$a_n(x) = \prod_{i=1}^{\ell} \left(x - \left(2 \cos \frac{m_i \pi}{h} \right) \right) .$$

Set $x = 2$.

$$\begin{aligned} a_n(2) &= \prod_{i=1}^{\ell} \left(2 - \left(2 \cos \frac{m_i \pi}{h} \right) \right) \\ &= 2^\ell \prod_{i=1}^{\ell} \left(1 - \cos \frac{m_i \pi}{h} \right) \\ &= 2^\ell \prod_{i=1}^{\ell} 2 \sin^2 \frac{m_i \pi}{2h} \\ &= 2^{2\ell} \prod_{i=1}^{\ell} \sin^2 \frac{m_i \pi}{2h} . \end{aligned}$$

To prove the formula we calculate $a_n(2)$. We have

$$p_n(x) = (x - \xi_1)(x - \xi_2) \dots (x - \xi_\ell) = (x - (4 - \xi_1))(x - (4 - \xi_2)) \dots (x - (4 - \xi_\ell)) .$$

This implies that $p_n(4) = \xi_1 \xi_2 \dots \xi_\ell = \det C$. Since $a_n(x) = p_n(x+2)$ we have

$$a_n(2) = p_n(4) = \det C .$$

□

Remark 9. *The formula can also be written in the form:*

$$2^\ell \prod_{i=1}^{\ell} \sin \frac{m_i \pi}{2h} = \sqrt{\det C} .$$

It is not clear what is the significance of the factor 2^ℓ . We note that the sum of the Betti numbers is 2^ℓ .

Remark 10. *One can compute $a_n(2)$ case by case using properties of Chebyshev polynomials:*

- (1) A_n $a_n(x) = U_n\left(\frac{x}{2}\right) \Rightarrow a_n(2) = U_n(1) = n+1 = \det C$.
- (2) B_n $a_n(x) = 2T_n\left(\frac{x}{2}\right) \Rightarrow a_n(2) = 2T_n(1) = 2 = \det C$.
- (3) C_n $a_n(x) = 2T_n\left(\frac{x}{2}\right) \Rightarrow a_n(2) = 2T_n(1) = 2 = \det C$.
- (4) D_n $a_n(x) = 2xT_n\left(\frac{x}{2}\right) \Rightarrow a_n(2) = 4T_n(1) = 4 = \det C$.
- (5) G_2 $a_2(x) = x^2 - 3 \Rightarrow a_2(2) = 1 = \det C$.
- (6) F_4 $a_4(x) = x^4 - 4x^2 + 1 \Rightarrow a_4(2) = 1 = \det C$.
- (7) E_6 $a_6(x) = x^6 - 5x^4 + 5x^2 - 1 \Rightarrow a_6(2) = 3 = \det C$.
- (8) E_7 $a_7(x) = x^7 - 6x^5 + 9x^3 - 3x \Rightarrow a_7(2) = 2 = \det C$.
- (9) E_8 $a_8(x) = x^8 - 7x^6 + 14x^4 - 8x^2 + 1 \Rightarrow a_8(2) = 1 = \det C$.

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